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Hydrodynamic characteristics of bodies in channels

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The effect of channel walls on the hydrodynamic characteristics of fixed or oscillating bodies is discussed using classical linear water wave theory. Particular attention is paid to the occurrence of trapped modes persisting local to the fixed body and which are manifested in a non-uniqueness of the corresponding forced problem at the trapped mode frequency. The general ideas are illustrated by consideration of two simple geometries for which semi-analytic solutions are available, namely a circular cylinder either partly immersed or extending throughout the water depth, and a thin vertical plate parallel to the channel walls which extends throughout the water depth. Conclusions are drawn concerning the conditions under which trapped modes may exist and their effect on the hydrodynamic characteristics of more general bodies.

1. Introduction

Experiments carried out to determine the hydrodynamic characteristics of bodies usually need to be carried out in wave tanks even though knowledge of the behaviour in the open sea is required. It is thus important to understand both qualitatively and quantitatively how the tank walls affect quantities such as the exciting force on a fixed body due to an incident wave or the added mass and damping coefficients for a body making small simple harmonic oscillations.

We consider the case of a channel of constant width $2d$ and constant depth h and we assume that the channel is infinitely long. We also assume that the fluid is incompressible and that the fluid motion is irrotational. Then the governing equation is Laplace's equation and it is well known that there is a countable set of discrete cut-off frequencies with n propagating modes possible when the frequency lies between the $(n-1)$th and the nth cut-off $(n \geqslant 1)$. The possible modes are alternately symmetric and antisymmetric about the centreline of the channel.

There are many important nonlinear aspects to the physical situation when the frequency is close to a cut-off value. For example Tulin & Yao (1992) have provided systematic data for the phenomenon of the generation of slowly modulated propagating wave groups at frequencies just below the first cut-off for symmetric modes. In this paper however we shall restrict our attention to linear theory. There are many situations in which the linear solution provides the dominant contribution to the hydrodynamic characteristics and despite being much simpler the nature of the linear solutions is still not fully understood.

A simple geometry that can be used to shed light on the various phenomena that can occur with wave–body interactions in a channel is the vertical circular cylinder, either extending throughout the depth or truncated. For the circular cylinder extending throughout the depth a vast literature exists since the scattering problem can be interpreted as a problem in acoustics. Most notable perhaps is the work of Twersky (1962), though most of the extensive hydrodynamic interest in the problem was generated by a paper by Spring & Monkmeyer (1975). Recently there has been much work on related problems. Thus the method of images has been used by, amongst others, Yeung & Sphaier (1989*a*,*b*) to solve the radiation problem for all modes of motion for a truncated cylinder immersed through the free surface. More recently Linton & Evans (1992*a*) and McIver & Bennett (1993) have shown how a multipole method can lead to simpler and easier to compute forms for the various quantities of interest in problems of this type.

Callan, Linton & Evans (1991) proved that for a vertical cylinder extending throughout the depth and placed on the centreline of the channel there exists a discrete mode below the first cut-off for the channel, antisymmetric about the centreline, which satisfies the condition of zero normal velocity on all solid boundaries and which has finite energy. They term such a mode a trapped mode. Evans (1992) proved that such antisymmetric trapped modes also exist for a vertical plate on the centreline extending all the way to the bottom and Evans, Linton & Ursell (1993) proved that these modes are still possible for an off-centre plate even though in this case antisymmetry cannot be imposed and propagating modes are possible at all frequencies.

Each of these problems has an acoustical counterpart and in particular the thin plate on the centreline of an acoustic waveguide has been shown experimentally by Parker (1966) to exhibit such modes which he terms acoustic resonances. There is also evidence for acoustic resonances in the circular cylinder case (Bearman & Graham 1980, pp. 231–232). A full review of the occurrence of acoustic resonances is given in Parker & Stoneman (1989).

Evans & Linton (1991) used numerical methods based on matched eigenfunction expansions to show that such modes exist for vertical cylinders of rectangular cross-section whilst Linton & Evans (1992*b*) have used the numerical solution of integral equations to show that such modes exist for a wide class of cross-sections, all symmetric about the channel centreline. In all the above cases the body extends throughout the fluid depth. Numerical evidence presented in Linton & Evans (1992*a*) suggests that such modes exist for truncated cylinders also.

These trapped modes are clearly related to the non-uniqueness of an associated forcing problem. For example, if we oscillate a cylinder which is placed on the centreline in sway at a trapped mode frequency, then since both the solution to the sway problem and the trapped mode are antisymmetric about the centreline we can add any multiple of the trapped mode to a solution to the sway problem without affecting the boundary conditions.

An illustration as to how this non-uniqueness affects the hydrodynamic coefficients of a circular cylinder on the centreline, in sway, together with the effect of the higher cut-off frequencies, was given in Linton & Evans (1992*a*). Their results showed that the trapped-mode frequency, which occurs below the first cut-off frequency and hence in a region where the damping coefficient is zero, corresponds to a singularity in the added mass coefficient. This is in contrast to the behaviour of these coefficients near higher cut-off frequencies where both coefficients exhibit spiky but non-singular behaviour.

In this paper we explore these phenomena in greater detail by considering radiation problems for different geometrical configurations, concentrating in particular on the behaviour of the hydrodynamic coefficients near to the trapped-mode and cut-off frequencies, so as to gain a thorough understanding of such problems.

We begin in §2 by first discussing again the circular cylinder on the centreline described in Linton & Evans (1992*a*) before solving the problem of the sway motion of an off-centre cylinder. This enables conclusions to be drawn concerning the different qualitative behaviour of the hydrodynamic characteristics of full bodies on and off the centreline.

In §3 we consider radiation problems for thin vertical plates aligned with the channel walls and show that there are fundamental differences between problems involving such thin bodies and the full bodies considered in §2.

In order to assist in our understanding of these problems we make use of various relations which exist between the solutions of scattering and radiation problems for bodies in channels. These relations, being an extension of those derived by Srokosz (1980) for problems symmetric about the centreline are derived in an Appendix.

2. Circular cylinders

In this section we consider the case of a vertical circular cylinder of radius a placed in a channel of width $2d$ and depth h. Axes (x, y, z) are chosen with x measured along the channel, y measured across the channel and z measured vertically upwards. The origin of the coordinate system lies on the centreline of the channel in the plane of the undisturbed free surface. The cylinder is centred at $(x, y) = (0, b)$, $(0 \leqslant b < d$, $0 < a \leqslant d - b)$, and for the most part we will be concerned with cylinders that extend throughout the water depth.

To begin with consider the case when the cylinder is on the centreline of the channel, i.e. $b = 0$. Two recent papers have examined problems with this geometry. In the first of these Callan *et al.* (1991) proved that trapped modes, modes of oscillation at a particular frequency which have finite energy, exist for cylinders of sufficiently small radius. Subsequently the scattering problem, together with the surge and sway radiation problems, was solved using multipole expansions in Linton & Evans (1992*a*)

Callan *et al.* (1991) proved that for sufficiently small cylinders there is at least one discrete wavenumber, given by $k = k^* < \pi/2d$, at which the homogeneous boundary value problem with zero normal velocity on all solid boundaries, $\phi = 0$ on $y = 0$, and $\phi \to 0$ as $|x| \to \infty$, has a non-trivial solution. Computations suggest that there is in fact one and only one such wavenumber for all sizes of cylinder $0 < a/d \leqslant 1$. This trapped mode solution is symmetric about $x = 0$. The condition $\phi = 0$ on $y = 0$, implying antisymmetry of the fluid motion about $y = 0$, ensures that no propagating waves can exist in the range $0 < 2kd < \pi$.

The existence of a trapped mode solution can also be seen from the solution to the sway radiation problem given in Linton & Evans (1992*a*). This problem is also antisymmetric about the centreline of the channel and so again no waves can exist below the first cut-off frequency. The symmetry of the problem implies that the added mass and damping matrices each have only one non-zero element (M_{22} and B_{22}) which after suitable non-dimensionalization we term μ and ν respectively. Below the first cut-off frequency, equation (A 33), which relates the sway damping coefficient to the energy radiated down the channel, implies that $\nu = 0$. Typical behaviour of the sway added mass in this range is shown in figure 1. Due to the existence of a trapped mode at $k = k^*$ the sway radiation problem does not have a unique solution at this value (since a multiple of the trapped mode can always be added to a solution) and this manifests itself as a singularity in the solution.

Above the first cut-off frequency $kd = \pi/2$ waves *are* generated and ν is no longer zero. At freqencies a little below the higher antisymmetric cut-off frequencies $kd = (n + 1/2)\pi$, $n = 1, 2, \ldots$, the added mass and damping coefficients exhibit spiky behaviour as shown in figure $2(a, b)$ for $n = 1$. The height of the spike in the damping coefficient and the total extent of the added mass spike can be seen to be very nearly

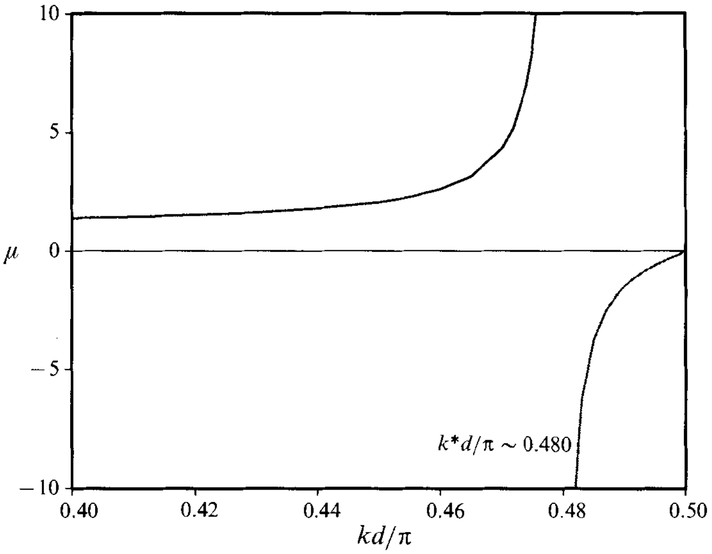

FIGURE 1. Sway added mass coefficient, μ, below the first cut-off frequency, for a cylinder on the centreline with $a/d = 0.3$, $a/h = 0.1$.

equal and this can be explained by the following argument, a brief description of which is given in Linton & Evans (1992*a*). The quantities ν and μ are the real and imaginary parts respectively of a complex force coefficient $q(\omega)$, say, and a large spike can be shown to correspond to a simple pole in the complex frequency plane close to the real axis. This pole must lie in the lower half-plane from causality considerations (Wehausen 1971). Near this pole, $\omega = \omega_0$ say, $q(\omega) \approx A/(\omega - \omega_0)$. Since the real ω-axis can be defined by the equation $|\omega - \overline{\omega_0}| = |\omega - \omega_0|$ it follows that as ω moves along the real axis close to the pole $\omega = \omega_0$,

$$\left| q + \frac{A}{\omega_0 - \overline{\omega_0}} \right| = \left| \frac{A}{\omega_0 - \overline{\omega_0}} \right|, \tag{2.1}$$

so that q moves round a circle, centre $A(\overline{\omega_0} - \omega_0)^{-1}$, radius $|A(\overline{\omega_0} - \omega_0)^{-1}|$. If we write $\omega_0 = a - \mathrm{i}\beta$, $\beta > 0$, then since the damping coefficient, and hence the real part of q, must always be greater than or equal to zero (from (A 33)), the centre of this circle must lie on the positive real axis. Thus we write $A = 2\mathrm{i}\alpha$, $\alpha > 0$, and we see that $\nu + \mathrm{i}\mu$ maps out a circle, centred at $\nu = \alpha/\beta$, with radius α/β. This is illustrated in figure 3. From the diagram it is clear that the total extent of the spikes in the added mass and damping coefficients both correspond to the diameter of this circle and are thus equal.

When the cylinder is not on the centreline of the channel, $b \neq 0$, the situation is fundamentally different. The scattering problem in this case has been solved by McIver & Bennett (1993). Here we extend the multipole method described in Linton & Evans (1992*a*). The boundary value problem we wish to solve is given by (A 2)–(A 5) together with a body boundary condition which in this case is

$$\frac{\partial \phi}{\partial r} = U \sin \theta, \tag{2.2}$$

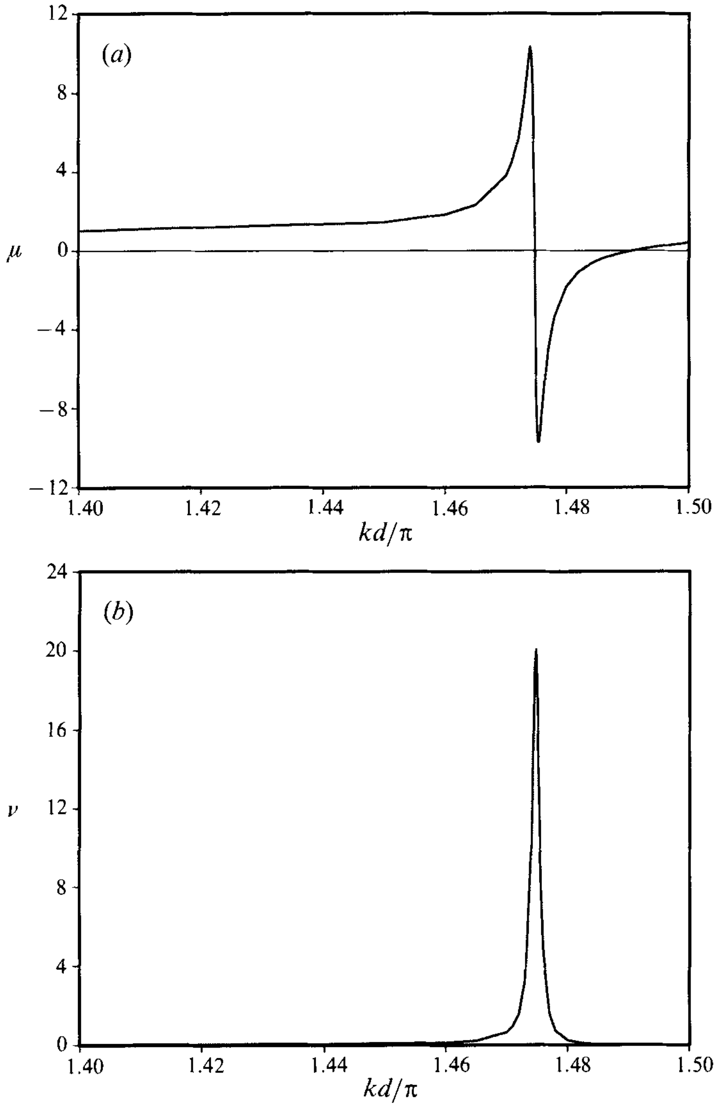

FIGURE 2. Sway hydrodynamic coefficients, below the second cut-off frequency, for a cylinder on the centreline with $a/d = 0.3$, $a/h = 0.1$. (a) Added mass, μ; (b) damping, ν.

where polar coordinates (r, θ) are defined by $x = r\cos\theta$, $y - b = r\sin\theta$. We also need a radiation condition of the form (A 19).

Unlike the case of the scattering problem the depth variation in radiation problems cannot be factored out. We thus define depth eigenfunctions

$$f_m(z) = N_m^{-\frac{1}{2}} \cos k_m(z+h), \qquad m = 0, 1, \ldots, \tag{2.3}$$

where

$$N_m = \frac{1}{2}\left(1 + \frac{\sin 2k_m h}{2k_m h}\right) \tag{2.4}$$

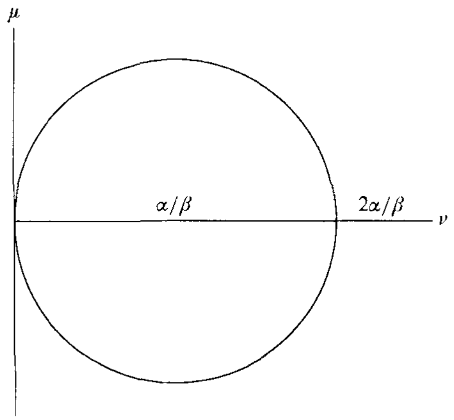

FIGURE 3. Path of the complex force coefficient $\nu + i\mu$ as ω moves along the real axis close to the pole $\omega = \omega_0$.

and k_m satisfies

$$k_m \tan k_m h + K = 0. \tag{2.5}$$

Here k_m, $m \geqslant 1$, are real and positive, whilst $k_0 = ik$, k real and positive. The functions $f_m(z)$, each of which satisfies the free-surface and bottom boundary conditions, satisfy the orthogonality relations

$$\frac{1}{h}\int_{-h}^{0} f_m(z) f_n(z)\,\mathrm{d}z = \delta_{mn}. \tag{2.6}$$

Next we define

$$\gamma(t) = \begin{cases} -\mathrm{i}(1-t^2)^{\frac{1}{2}}, & t \leqslant 1 \\ (t^2-1)^{\frac{1}{2}}, & t > 1, \end{cases} \tag{2.7}$$

$$c_{2n}(t) = \begin{cases} \cos[2n \sin^{-1} t], & t \leqslant 1 \\ (-1)^n \cosh[2n \cosh^{-1} t], & t > 1, \end{cases} \tag{2.8}$$

$$c_{2n+1}(t) = \begin{cases} \cos[(2n+1) \sin^{-1} t], & t \leqslant 1 \\ \mathrm{i}(-1)^n \sinh[(2n+1) \cosh^{-1} t], & t > 1, \end{cases} \tag{2.9}$$

$$s_{2n}(t) = \begin{cases} \sin[2n \sin^{-1} t], & t \leqslant 1 \\ -\mathrm{i}(-1)^n \sinh[2n \cosh^{-1} t], & t > 1, \end{cases} \tag{2.10}$$

$$s_{2n+1}(t) = \begin{cases} \sin[(2n+1) \sin^{-1} t], & t \leqslant 1 \\ (-1)^n \cosh[(2n+1) \cosh^{-1} t], & t > 1. \end{cases} \tag{2.11}$$

With these definitions, M defined by (A 15), and t_p, $p = 0, \ldots, M$ defined by (A 14), we can derive polar coordinate expansions for channel multipoles centred at $(x, y) = (0, b)$ and symmetric about $x = 0$. We obtain

$$\phi_{2n,0} = H_{2n}(kr)\cos 2n\theta + \sum_{q=0}^{\infty}\{E\{2q, 2n; 0\} J_{2q}(kr) \cos 2q\theta$$
$$+ E\{2q+1, 2n; 0\} J_{2q+1}(kr) \sin(2q+1)\theta\}, \tag{2.12}$$

$$\sim \frac{1}{2kd}\sum_{p=0}^{M}\frac{\epsilon_p}{t_p}\left\{\cos\left[\frac{p\pi}{2d}(y-b)\right]\right.$$

$$\left.+(-1)^p\cos\left[\frac{p\pi}{2d}(y+b)\right]\right\}e^{\pm ikxt_p}c_{2n}(t_p), \qquad x\to\pm\infty, \tag{2.13}$$

$$\phi_{2n+1,0} = H_{2n+1}(kr)\sin(2n+1)\theta + \sum_{q=0}^{\infty}\{E\{2q,2n+1;0\}J_{2q}(kr)\cos 2q\theta$$

$$+E\{2q+1,2n+1;0\}J_{2q+1}(kr)\sin(2q+1)\theta\}, \tag{2.14}$$

$$\sim \frac{1}{kd}\sum_{p=0}^{M}\frac{1}{t_p}\left\{\sin\left[\frac{p\pi}{2d}(y-b)\right]\right.$$

$$\left.-(-1)^p\sin\left[\frac{p\pi}{2d}(y+b)\right]\right\}e^{\pm ikxt_p}c_{2n+1}(t_p), \qquad x\to\pm\infty, \tag{2.15}$$

whilst for $m \geqslant 1$,

$$\phi_{2n,m} = K_{2n}(k_m r)\cos 2n\theta + \sum_{q=0}^{\infty}\{E\{2q,2n;m\}I_{2q}(k_m r)\cos 2q\theta$$

$$+E\{2q+1,2n;m\}I_{2q+1}(k_m r)\sin(2q+1)\theta\}, \tag{2.16}$$

$$\phi_{2n+1,m} = K_{2n+1}(k_m r)\sin(2n+1)\theta + \sum_{q=0}^{\infty}\{E\{2q,2n+1;m\}I_{2q}(k_m r)\cos 2q\theta$$

$$+E\{2q+1,2n+1;m\}I_{2q+1}(k_m r)\sin(2q+1)\theta\}, \tag{2.17}$$

where

$$E\{2q,2n;0\} = -\frac{2i\epsilon_q}{\pi}\,⨍_0^\infty abc\frac{e^{-2k\gamma d}+\cosh 2k\gamma b}{\gamma\sinh 2k\gamma d}c_{2q}(t)c_{2n}(t)\,dt, \tag{2.18}$$

$$E\{2q+1,2n;0\} = -\frac{4}{\pi}\,⨍_0^\infty abc\frac{\sinh 2k\gamma b}{\gamma\sinh 2k\gamma d}c_{2q+1}(t)c_{2n}(t)\,dt, \tag{2.19}$$

$$E\{2q,2n+1;0\} = -\frac{2\epsilon_q}{\pi}\,⨍_0^\infty abc\frac{\sinh 2k\gamma b}{\gamma\sinh 2k\gamma d}c_{2q}(t)c_{2n+1}(t)\,dt, \tag{2.20}$$

$$E\{2q+1,2n+1;0\} = -\frac{4i}{\pi}\,⨍_0^\infty abc\frac{e^{-2k\gamma d}-\cosh 2k\gamma b}{\gamma\sinh 2k\gamma d}c_{2q+1}(t)c_{2n+1}(t)\,dt, \tag{2.21}$$

$$E\{2q,2n;m\} = \epsilon_q\int_1^\infty \frac{e^{-2k_m dt}+\cosh 2k_m bt}{\gamma\sinh 2k_m dt}c_{2q}(t)c_{2n}(t)\,dt, \tag{2.22}$$

$$E\{2q+1,2n;m\} = -2\int_1^\infty \frac{\sinh 2k_m bt}{\gamma\sinh 2k_m dt}s_{2q+1}(t)c_{2n}(t)\,dt, \tag{2.23}$$

$$E\{2q,2n+1;m\} = \epsilon_q\int_1^\infty \frac{\sinh 2k_m bt}{\gamma\sinh 2k_m dt}c_{2q}(t)s_{2n+1}(t)\,dt, \tag{2.24}$$

$$E\{2q+1,2n+1;m\} = 2\int_1^\infty \frac{e^{-2k_m dt}-\cosh 2k_m bt}{\gamma\sinh 2k_m dt}s_{2q+1}(t)s_{2n+1}(t)\,dt. \tag{2.25}$$

We write

$$\phi(x,y,z) = Ua\sum_{m=0}^{\infty}f_m(z)\sum_{n=0}^{\infty}\alpha_{n,m}\phi_{n,m} \tag{2.26}$$

for some unknowns $\alpha_{n,m}$. The boundary condition on the cylinder, (2.2), can be written

$$\frac{\partial\phi}{\partial r} = U\sin\theta\sum_{m=0}^{\infty}f_m(z)F_m, \tag{2.27}$$

where

$$F_m = \frac{1}{h}\int_{-h}^{0} f_m(z)\,\mathrm{d}z = N_m^{-\frac{1}{2}}\frac{\sin k_m h}{k_m h}. \tag{2.28}$$

Applying this boundary condition and then using the orthogonality of the depth eigenfunctions and the trigonometric functions leads to an infinite set of infinite systems of equations:

$$\sum_{n=0}^{\infty}[\delta_{qn} + E\{q,n;m\}\mathscr{J}'_{q,m}(a)/\mathscr{H}'_{q,m}(a)]\alpha_{n,m} = \delta_{q1}F_m/a\mathscr{H}'_{1,m}(a), \qquad q,m \geqslant 0, \tag{2.29}$$

where

$$\mathscr{H}_{q,m}(r) = \begin{cases} H_q(kr), & m=0 \\ K_q(k_m r), & m \geqslant 1, \end{cases} \tag{2.30}$$

and

$$\mathscr{J}_{q,m}(r) = \begin{cases} J_q(kr), & m=0 \\ I_q(k_m r), & m \geqslant 1. \end{cases} \tag{2.31}$$

The added mass and damping coefficients, non-dimensionalized with respect to the mass of fluid displaced by the cylinder are then given, from (A 24), by

$$\mu + \mathrm{i}\nu = \frac{F_0}{kaJ_1'(ka)}\left[J_1(ka)F_0 - \frac{2\mathrm{i}}{\pi}\alpha_{1,0}\right] - \sum_{m=1}^{\infty}\frac{F_m}{k_m a I_1'(k_m a)}\left[\alpha_{1,m} + I_1(k_m a)F_m\right]. \tag{2.32}$$

We note that since the fluid motion is not antisymmetric about $y=0$ there is wave radiation at all frequencies. Below the first cut-off, $kd = \pi/2$, we now have behaviour similar to that near the higher cut-off frequencies in the $b=0$ case, with no singularity in the added mass, and there do not appear to be any trapped modes. Figure 4(*a*, *b*) shows that as $b/d \to 0$ the initial spikes in μ and ν, computed from (2.32), get higher and narrower as well as occurring at smaller values of kd, and this corresponds to the value of β, introduced in the discussion prior to (2.3), becoming smaller. Thus as $b/d \to 0$ the pole below the real axis, near to the first cut-off frequency, moves toward this axis and when $b=0$ it reaches the axis, $\beta = 0$, and instead of moving in a circle $q \equiv \nu + \mathrm{i}\mu$ moves up the imaginary axis to $\mu = +\infty$ and then returns along this axis from $\mu = -\infty$ with $\nu = 0$ throughout. This provides an explanation for the singular behaviour of the sway added mass coefficient in the $b=0$ case, below $kd = \pi/2$.

Another quantity which provides insight into the nature of the solutions to these problems is the cross-channel exciting force on the cylinder due to an incident plane wave. From (A 44) we see that this is proportional to the amplitude of the fundamental mode that is radiated when the cylinder moves in sway. By symmetry this force is zero when $b=0$ for all values of kd and so no fundamental mode is radiated. When $b \neq 0$ results from McIver & Bennett (1993) show that this force is, as one would expect, no longer zero and in fact exhibits very spiky behaviour which is directly related through (A 44) and (A 33) to the damping coefficient.

We would expect the above conclusions to be valid for truncated cylinders also. Radiation and scattering problems for two types of truncated cylinders were considered in Linton & Evans (1992*a*). The analysis required to solve these problems is considerably more involved than for the non-truncated case. It is clear from their results that for the case of a cylinder immersed through the free surface but only extending part way to the bottom, the qualitative behaviour of the hydrodynamic quantities is the same as for the non-truncated case, with significant quantitative differences only in very long waves or when the draught of the cylinder is very small.

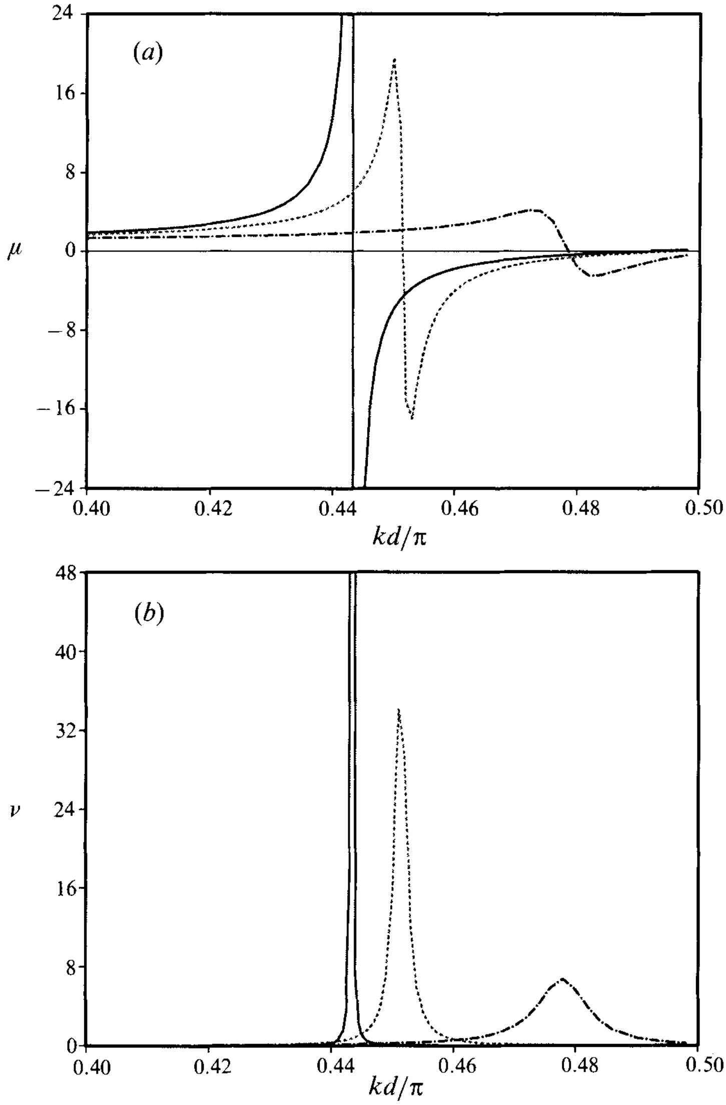

FIGURE 4. Sway hydrodynamic coefficients, below the first cut-off frequency, for three off-centre cylinders with $a/d = 0.5$, $a/h = 0.1$. ——, $b/d = 0.05$; - - - - -, $b/d = 0.2$; —·—, $b/d = 0.4$. (*a*) Added mass, μ; (*b*) damping, ν.

Computations suggest that a singularity is again present in the sway added mass in the range $0 < kd < \pi/2$ and thus that trapped modes exist for this geometry also. As the draught of the cylinder increases up to the total water depth the various hydrodynamic quantities tend rapidly and continuously to the non-truncated values, the only exception being for the heave motion of a truncated cylinder as there is no equivalent problem in the non-truncated case. There is no reason to suppose that when $b \neq 0$ the effect of such truncation would be any greater.

For a bottom-mounted cylinder not extending through the free surface some of the qualitative features present in the non-truncated case are retained but quantitatively the results are very different. Again a trapped mode appears to be present just below the first antisymmetric cut-off and there is spiky behaviour in the hydrodynamic characteristics near the relevant cut-off frequencies for each radiation problem. In this case however there is no continuous change towards the non-truncated values as the height of the cylinder approaches the water depth, due to the fundamental difference between a problem where the body intersects the free surface and one where it does not.

Variations in the hydrodynamic characteristics of a vertical circular cylinder extending throughout the depth of the fluid as the radius to channel semi-width ratio, a/d, varies are well documented in the papers cited above. Some general comments are appropriate here. One would expect the effects of the channel walls to diminish as a/d becomes small, either for fixed kd or for fixed ka. This is indeed the case for the exciting force on the cylinder as results in Linton & Evans (1992*a*) and McIver & Bennett (1993) demonstrate. However work by Thomas (1991) and McIver (1992) shows that the influence of the channel walls on the pressure values on the cylinder is considerable and their values vary in an unpredictable way and tend only very slowly to the open sea values. In the case of the added mass and damping coefficients it appears that increasing a/d decreases the spiky behaviour near to the cut-off frequencies. This can be explained as follows. The spiky response in the hydrodynamic force on the cylinder is due to the fact that in the absence of the cylinder a resonant 'sloshing' mode is possible at each cut-off frequency. The presence of the cylinder inhibits this resonant mode and the larger the cylinder the more it can suppress the spiky behaviour.

3. Plates

The case of a thin vertical plate aligned with the channel walls gives rise to some very different behaviour. For the case of the plate on the centreline extending throughout the depth, the existence of trapped modes was suggested, using partly analytical, partly numerical arguments, by Evans & Linton (1991) and subsequently proved rigorously by Evans (1992) for sufficiently long plates. Much earlier Parker (1966, 1967) had demonstrated experimentally the existence of these modes which he called acoustic resonances, and had computed their frequencies using a fully numerical method. Unaware of this work, in their paper in 1991, Evans & Linton used the method of matched eigenfunction expansions to compute the trapped mode wavenumbers and their results indicate that a trapped mode symmetric about $x = 0$ exists for all values of $a/d > 0$. As a/d increases more modes are possible, modes antisymmetric and symmetric about $x = 0$ appearing in turn each time a/d passes through an integer value. Thus for $n - 1 < a/d < n$ $(n \geqslant 1)$, n trapped modes are possible.

The sway radiation problem for such a body can be solved using the same matched eigenfunction method. Since we shall also be interested in plates off the centreline we shall first solve this more general problem with $b \neq 0$ and then indicate the changes required to recover the corresponding results for $b = 0$. Notice that these radiation problems are symmetric about $x = 0$. Thus we seek to solve the following boundary value problem for $\phi(x, y, z)$:

$$\nabla^2 \phi = 0, \qquad x > 0, -d < y < d, -h < z < 0, \tag{3.1}$$

$$K\phi = \frac{\partial \phi}{\partial z} \qquad \text{on } z = 0, \tag{3.2}$$

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{on } z = -h, \tag{3.3}$$

$$\frac{\partial \phi}{\partial y} = 0 \qquad \text{on } y = \pm d, \tag{3.4}$$

$$\frac{\partial \phi}{\partial y} = 1 \qquad \text{on } y = b,\ 0 < x < a,\ |b| < d, \tag{3.5}$$

$$\frac{\partial \phi}{\partial x} = 0 \qquad \text{on } x = 0. \tag{3.6}$$

We split the fluid region into three parts. Region 1 is $\{0 < x < a, b < y < d, -h < z < 0\}$, region 2 is $\{0 < x < a, -d < y < b, -h < z < 0\}$ and region 3 is $\{x > a, -d < y < d, -h < z < 0\}$. Writing $c = d - b$, $f = d + b$, orthogonal cross-channel eigenfunctions suitable for these three regions are, with $n = 0, 1, \dots$,

$$\psi_n^{(1)}(y) = \epsilon_n^{\frac{1}{2}} \cos(v_n(d - y)), \qquad v_n = n\pi/c, \tag{3.7}$$

$$\psi_n^{(2)}(y) = \epsilon_n^{\frac{1}{2}} \cos(\mu_n(d + y)), \qquad \mu_n = n\pi/f, \tag{3.8}$$

and

$$\psi_n^{(3)}(y) = \epsilon_n^{\frac{1}{2}} \cos(\lambda_n(d - y)), \qquad \lambda_n = n\pi/2d. \tag{3.9}$$

Let ϕ in region i be ϕ_i, $i = 1, 2, 3$, and write

$$\phi_1 = \sum_{m=0}^{\infty} f_m(z) \left[-\frac{F_m \cosh k_m(d - y)}{k_m \sinh k_m c} + \sum_{n=0}^{\infty} U_{nm}^{(1)} \frac{\cosh \alpha_{nm} x}{\alpha_{nm} \sinh \alpha_{nm} a} \psi_n^{(1)}(y) \right], \tag{3.10}$$

$$\phi_2 = \sum_{m=0}^{\infty} f_m(z) \left[\frac{F_m \cosh k_m(d + y)}{k_m \sinh k_m f} + \sum_{n=0}^{\infty} U_{nm}^{(2)} \frac{\cosh \beta_{nm} x}{\beta_{nm} \sinh \beta_{nm} a} \psi_n^{(2)}(y) \right], \tag{3.11}$$

$$\phi_3 = \sum_{m=0}^{\infty} f_m(z) \sum_{n=0}^{\infty} U_{nm}^{(3)} \frac{e^{-\gamma_{nm}(x-a)}}{-\gamma_{nm}} \psi_n^{(3)}(y), \tag{3.12}$$

where

$$\alpha_{nm} = (v_n^2 + k_m^2)^{\frac{1}{2}}, \qquad \beta_{nm} = (\mu_n^2 + k_m^2)^{\frac{1}{2}}, \qquad \gamma_{nm} = (\lambda_n^2 + k_m^2)^{\frac{1}{2}},$$

$f_m(z)$ and k_m are defined by (2.3)–(2.5), and F_m is defined by (2.28). With these definitions, (3.1)–(3.6) are all satisfied and the unknown coefficients $U_{nm}^{(i)}$, $i = 1, 2, 3$, can be found by matching ϕ_i and $\partial \phi_i / \partial x$ across $x = a$. Writing

$$d_{mn} = \frac{1}{c} \int_b^d \psi_m^{(1)}(y) \psi_n^{(3)}(y)\, \mathrm{d}y, \tag{3.13}$$

$$e_{mn} = \frac{1}{f} \int_{-d}^b \psi_m^{(2)}(y) \psi_n^{(3)}(y)\, \mathrm{d}y \tag{3.14}$$

we obtain

$$U_{qm}^{(1)} = \sum_{n=0}^{\infty} U_{nm}^{(3)} d_{qn}, \qquad U_{qm}^{(2)} = \sum_{n=0}^{\infty} U_{nm}^{(3)} e_{qn} \tag{3.15}$$

and

$$\sum_{n=0}^{\infty} \left[\delta_{qn} + \gamma_{qm} \sum_{r=0}^{\infty} \left(c d_{rn} d_{rq} \frac{\coth \alpha_{rm} a}{2\alpha_{rm} d} + f e_{rn} e_{rq} \frac{\coth \beta_{rm} a}{2\beta_{rm} d} \right) \right] U_{nm}^{(3)}$$

$$= \frac{\epsilon_q^{\frac{1}{2}} F_m}{2k_m d \gamma_{qm}} \lambda_q \sin \lambda_q c\, (\coth k_m c + \coth k_m f), \qquad q, m \geqslant 0. \tag{3.16}$$

Noting that

$$d_{00} = e_{00} = 1, \qquad d_{r0} = e_{r0} = 0, \quad r \geqslant 1, \qquad cd_{0n} = -fe_{0n}, \quad n \geqslant 1,$$

it can be shown that the $q = m = 0$ equation is simply

$$U_{00}^{(3)}(1 - \mathrm{i}\cot ka) = 0 \tag{3.17}$$

and hence

$$U_{00}^{(3)} = 0, \tag{3.18}$$

showing that no waves of the form $\exp(\mathrm{i}kx)$ are radiated away from the plate to $x = \infty$. In fact the $m = 0$ system of equations from (3.16) reduces to

$$\sum_{n=1}^{\infty}\left[\delta_{qn} + \gamma_{q0}\sum_{r=0}^{\infty}\left(cd_{rn}d_{rq}\frac{\coth\alpha_{r0}a}{2\alpha_{r0}d} + fe_{rn}e_{rq}\frac{\coth\beta_{r0}a}{2\beta_{r0}d}\right)\right]U_{n0}^{(3)}$$

$$= -\frac{\epsilon_q^{\frac{1}{2}}F_0}{2kd\gamma_{q0}}\lambda_q\sin\lambda_q c\,(\cot kc + \cot kf), \qquad q \geqslant 1, \tag{3.19}$$

and it can be seen that provided $2kd < \pi$ this is a real system since γ_{n0} is real for all $n \geqslant 1$.

Non-dimensional added mass and damping coefficients are then given by

$$\mu + \mathrm{i}\nu = \frac{\omega M + \mathrm{i}B}{2\rho\omega adh}$$

$$= -\sum_{m=0}^{\infty}F_m\left[\frac{F_m}{k_m d}(\coth k_m c + \coth k_m f) - \sum_{n=0}^{\infty}(-1)^n\epsilon_n^{\frac{1}{2}}\left(\frac{U_{nm}^{(1)}}{\alpha_{nm}^2 ad} - \frac{U_{nm}^{(2)}}{\beta_{nm}^2 ad}\right)\right]. \tag{3.20}$$

Results for the $b = 0$ case can be obtained from the above analysis simply by letting $b \to 0$ throughout. However the $b = 0$ problem can be solved in a much simpler manner by taking account of the antisymmetry of the problem about $y = 0$. Thus we need only consider the region $y \geqslant 0$ and apply the condition $\phi = 0$ on $y = 0$, $x > a$.

Region 1 is as before, region 2 no longer exists and region 3 becomes $\{x > a, 0 < y < d, -h < z < 0\}$. Orthogonal cross-channel eigenfunctions suitable for region 3 are now

$$\psi_n^{(3)}(y) = 2^{\frac{1}{2}}\sin((n + \tfrac{1}{2})\pi y/d), \qquad n \geqslant 0. \tag{3.21}$$

The expansions for ϕ_1 and ϕ_3 are given by (3.10) and (3.12) as before where now

$$\gamma_{nm} = ((n + \tfrac{1}{2})^2\pi^2/d^2 + k_m^2).$$

Owing to the simplified nature of the geometry we can now proceed in two different ways. First we could obtain a system of equations for the unknowns $U_{nm}^{(3)}$ similar to (3.16) but we now also have the option of obtaining a system of equations for the unknowns $U_{nm}^{(1)}$ and since these are required in the computation of forces on the plate this is a more sensible approach. Thus we obtain

$$\sum_{n=0}^{\infty}\left[\delta_{qn} + \alpha_{qm}\tanh\alpha_{qm}a\sum_{r=0}^{\infty}\gamma_{rm}^{-1}d_{qr}d_{nr}\right]U_{nm}^{(1)} = \epsilon_q^{\frac{1}{2}}(-1)^q F_m\frac{\tanh\alpha_{qm}a}{\alpha_{qm}d}, \qquad q, m \geqslant 0, \tag{3.22}$$

which, like (3.19), is a real system provided $2kd < \pi$. Non-dimensional added mass

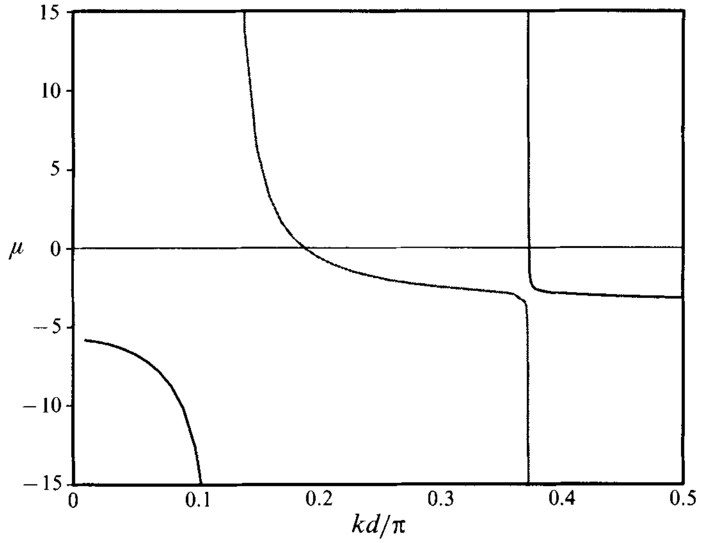

FIGURE 5. Sway added mass coefficient, μ, below the first cut-off frequency, for a plate on the centreline with $a/d = 3.5$, $a/h = 0.5$.

and damping coefficients are then given by

$$\mu + \mathrm{i}\nu = \frac{\omega M + \mathrm{i}B}{2\rho\omega a d h} = -2\sum_{m=0}^{\infty} F_m \left[\frac{F_m}{k_m d}\coth k_m d - \sum_{n=0}^{\infty} \frac{\epsilon_n^{\frac{1}{2}}(-1)^n U_{nm}^{(1)}}{\alpha_{nm}^2 a d}\right]. \tag{3.23}$$

Note that the antisymmetry of the solution about $y = 0$ has been used and the potential integrated along both sides of the plate.

Figure 5 shows the non-dimensional added mass coefficient, computed from (3.23), for the case of a plate on the centreline with $a/d = 3.5$, $a/h = 0.5$, over the range $0 < 2kd < \pi$. As in the case of the circular cylinder we find that in this range the damping coefficient is zero (as it must be since no waves are radiated) and the added mass is singular at those values of kd which correspond to trapped mode wavenumbers symmetric about $x = 0$. In the case of figure 5 the method of Evans & Linton (1991) gives two symmetric trapped mode wavenumbers at $kd/\pi \approx 0.126, 0.372$. (The antisymmetric trapped mode wavenumbers correspond to singularities in the added mass for a radiation problem which is itself antisymmetric about $x = 0$, for example a yawing plate with $\partial\phi/\partial y = x/a$ on $y = 0$.) Above the first cut-off frequency, computations show that the general form of the added mass and damping curves is the same as for the circular cylinder case discussed in the previous section, with spikes occurring in the hydrodynamic coefficients which can again be explained as a consequence of a pole close to the real axis in the complex force coefficient, $q(\omega)$.

When $b \neq 0$ however, the plate and the circular cylinder give rise to fundamentally different behaviour. Thus Evans *et al.* (1993) have proved that trapped modes exist for a plate off the centreline in the range $0 < 2kd < \pi$, wavenumbers for which in this case propagating modes would appear to be possible since there is no longer a condition of antisymmetry about $y = 0$. In fact for this problem it is clear that $\exp(\pm \mathrm{i}kx)$ is an eigenfunction with any $k \geqslant 0$ the corresponding eigenvalue. For this reason these trapped modes are said to be embedded in the continuous spectrum.

In all the previous examples trapped modes have given rise to singularities in the added mass which, since these are due to a singularity on the real axis in $q(\omega)$, can only occur if the damping coefficient is zero. This follows from the relation $q(\omega) \equiv \nu + \mathrm{i}\mu \approx 2\mathrm{i}\alpha/(\omega - \omega_0)$, α real, since if μ is singular ω_0 is real and hence ν vanishes. It can be seen from (3.18) and (A 33) that the damping coefficient is indeed zero over the range $0 < 2kd < \pi$ since the amplitude of the fundamental radiated mode (i.e. $\exp(\mathrm{i}k|x|)$) is zero for *all* values of k.

This remarkably simple result arising quite naturally from the general formulation above in terms of eigenfunction expansions should be capable of a simple explanation. The simplest way of proving the result is to use the extension of the Haskind relations (A 44). Since there is no cross-channel exciting force on the thin plate when a plane wave is incident on it, it follows, from (A 44), that the coefficient multiplying $\exp(\pm\mathrm{i}kx)$ in the far field of the sway radiation problem is zero.

This result is in fact true in more general problems than simply the sway problem. Consider the radiation problem for an off-centre plate parallel to the tank walls with the boundary condition $\partial\phi/\partial y = f(x)$ on $y = b, |x| < a$. Green's theorem applied to ϕ and another harmonic function ψ with $\partial\psi/\partial y = 0$ on $y = \pm d$ gives

$$\int_{S_\mathrm{B}+S_\infty} \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) \mathrm{d}S = 0,$$

with S_B and S_∞ as in (A 27). Taking $\psi = \exp(\pm\mathrm{i}kx)$ we see that $\partial\psi/\partial y = 0$ on S_B and ψ and $\partial\psi/\partial y$ are continous across the plate. Thus the integrals along each side of the plate cancel out and if

$$\phi \sim H(z)\sum_{n=0}^{M} A_n^{\pm} Y_n(y)\mathrm{e}^{\pm\mathrm{i}kxt_n} \qquad \text{as } x \to \pm\infty,$$

in the notation of the Appendix, the integrals over S_∞ show that $A_0^{\pm} = 0$. Thus any combination of N vertical plates aligned with the channel walls, moving such that $\partial\phi/\partial y = f_i(x)$ on the ith plate, $i = 1, \ldots, N$, will generate no waves in the range $0 < 2kd < \pi$. The above arguments suggest that trapped modes exist for any such configuration of fixed vertical plates.

Computations of the added mass coefficient from (3.20) show that as expected, this coefficient exhibits singular behaviour at values of kd corresponding to the trapped mode wavenumbers as computed by Evans *et al.* (1993).

4. Discussion

Some fairly general conclusions can be drawn from the above results.

The existence of trapped modes clearly implies the non-uniqueness of certain radiation problems since any multiple of the trapped mode can be added to the solution without affecting the boundary conditions. This non-uniqueness appears as a singularity in the added mass coefficient in a region where the damping coefficient is identically zero and this has been explained in terms of a pole of the complex force coefficient on the real axis. Thus it appears that trapped modes only exist in a region where the damping coefficient is zero. Note that this does not apply to all radiation problems: the problem of a heaving truncated cylinder on the centreline has a unique solution at the trapped mode frequency for the corresponding fixed body since the solution to the heave problem is symmetric about the centreline whereas the trapped mode is antisymmetric about this line.

Now any body symmetric about the channel centreline moving in sway will produce a fluid motion antisymmetric about the centreline so that if $0 < 2kd < \pi$ no waves will be radiated and the damping coefficient will indeed be identically zero over this range. In all such cases considered up to now trapped modes have been found (except in the case of a vertical plate perpendicular to the channel walls where the body does not interfere with the sloshing modes of the tank) and it seems reasonable to suggest that trapped modes exist for all such bodies.

In this paper we have shown that another problem where the damping coefficient vanishes over a range of values of kd is that of a swaying vertical plate aligned with the channel walls placed anywhere in the channel, or indeed any number of such bodies, and this explains why Evans *et al.* (1993) were able to find trapped modes in this case also.

Since the damping coefficient in sway is related through (A 33) and (A 44) to the cross-channel exciting force on the body due to an incident wave from infinity our arguments suggest that if this force is zero over a range of frequencies then a trapped mode will exist. From symmetry considerations this is always the case for a body symmetric about the channel centreline and it is also clearly true for thin vertical plates aligned with the channel walls. It seems unlikely that the cross-channel exciting force on any body which is not symmetric about the channel centreline will be zero over a range of frequencies and so in turn it seems unlikely that trapped modes will exist for such bodies.

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Appendix. General relations for bodies in channels

In the linear theory of interactions between water waves and bodies a number of general relations exist involving the various hydrodynamic quantities that arise. These relations were discovered over a number of years and a systematic derivation for the case of a single body in both two and three dimensions is given in Newman (1976). Srokosz (1980) extended these results to the case of a vertical cylinder placed on the centreline of a channel extending throughout the water depth with the property that the cross-section of the cylinder and any body motions are symmetric about the centreline. Here we shall derive relations suitable for any body in a channel moving in any mode of motion.

We make the usual assumptions of an incompressible, inviscid fluid and irrotational motion and we further assume that all motion is time-harmonic with angular frequency ω. Using linear water-wave theory we can define a velocity potential

$$\Phi(x,y,z,t) = \mathrm{Re}\{\phi(x,y,z)\mathrm{e}^{-\mathrm{i}\omega t}\} \tag{A 1}$$

which satisfies

$$\nabla^2\phi = 0 \qquad \text{in the fluid,} \tag{A 2}$$

$$K\phi = \frac{\partial \phi}{\partial z} \qquad \text{on } z = 0, \qquad K \equiv \omega^2/g, \tag{A 3}$$

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{on } z = -h, \tag{A 4}$$

and

$$\frac{\partial \phi}{\partial y} = 0 \qquad \text{on } y = \pm d. \tag{A 5}$$

In order to completely specify a boundary value problem for a wave–body interaction we must have a body boundary condition and suitable radiation conditions as $|x| \to \infty$.

First we will consider the body boundary condition. The wetted part of the body boundary will be denoted by S_B. If we restrict our attention to rigid body motions then the normal velocity of a point on the body surface can be written

$$V(t) = \mathrm{Re}\{[(U_1, U_2, U_3) + (U_4, U_5, U_6) \times \boldsymbol{r}].\boldsymbol{n}\mathrm{e}^{-\mathrm{i}\omega t}\}, \tag{A 6}$$

where U_1, U_2, U_3 are the components of surge, sway and heave and U_4, U_5, U_6 are the components of roll, pitch and yaw. The vector $\boldsymbol{r} \equiv ((x - x_0), (y - y_0), (z - z_0))$ is the position vector of the point on the body surface with the origin at (x_0, y_0, z_0), the centre of rotation, and $\boldsymbol{n} \equiv (n_1, n_2, n_3)$ is the unit outward normal from the body. If we define $(n_4, n_5, n_6) = \boldsymbol{r} \times \boldsymbol{n}$ then the linearized body boundary condition is

$$\frac{\partial \phi}{\partial n} = \sum_{i=1}^{6} U_i n_i \qquad \text{on } S_B. \tag{A 7}$$

For a general problem, ϕ can be decomposed into an incident potential, ϕ_I, a diffracted potential, ϕ_D, and six radiation potentials, ϕ_i, one for each mode of motion. Thus

$$\phi = \phi_I + \phi_D + \sum_{i=1}^{6} U_i \phi_i. \tag{A 8}$$

If we write $\phi_S = \phi_I + \phi_D$ then

$$\frac{\partial \phi_S}{\partial n} = 0 \qquad \text{on } S_B \tag{A 9}$$

and

$$\frac{\partial \phi_i}{\partial n} = n_i, \qquad i = 1, \ldots, 6, \qquad \text{on } S_B. \tag{A 10}$$

The correct far-field behaviour can be obtained by solving for ϕ in the absence of a body. Thus if we separate the variables and write $\phi(x, y, z) = X(x)Y(y)Z(z)$ we obtain

$$Z(z) = \cosh k(z + h), \tag{A 11}$$

where k is the unique positive solution of the dispersion relation $k \tanh kh = K$,

$$Y(y) = Y_n(y) = \begin{cases} \cos \lambda_n y, & n \text{ even}, \\ \sin \lambda_n y, & n \text{ odd}, \end{cases} \qquad n = 0, 1, \ldots, \tag{A 12}$$

where $\lambda_n = n\pi/2d$. Solutions for X are given by

$$X(x) = X_n(x) = \mathrm{e}^{\pm \mathrm{i}kxt_n} \tag{A 13}$$

where

$$t_n = \left[1 - \left(\frac{\lambda_n}{k}\right)^2\right]^{\frac{1}{2}}. \tag{A 14}$$

For the solution to behave like a propagating wave as $|x| \to \infty$, t_n must be real. We define an integer M by

$$M\pi < 2kd < (M + 1)\pi \tag{A 15}$$

and then for $n = 0, \ldots, M$, X_n represents a propagating wave. The frequencies corresponding to $2kd = m\pi$, m a positive integer, are called the cut-off frequencies for the channel.

An incident plane wave from $x = -\infty$ can thus be written as

$$\phi_{\rm I}^{(1)} = H(z)\mathrm{e}^{\mathrm{i}kx}, \tag{A 16}$$

where

$$H(z) = -\frac{\mathrm{i}gA\cosh k(z+h)}{\omega\cosh kh}. \tag{A 17}$$

Here A is the (assumed real) amplitude of the incident wave. Similarly an incident plane wave from $x = +\infty$ can be written as

$$\phi_{\rm I}^{(2)} = H(z)\mathrm{e}^{-\mathrm{i}kx}. \tag{A 18}$$

The far-field behaviour of the various potentials that appear in (A 8) is then

$$\phi_i \sim H(z)\sum_{n=0}^{M} A_{in}^{\pm} Y_n(y)\mathrm{e}^{\pm \mathrm{i}kxt_n} \qquad \text{as } x \to \pm\infty, \tag{A 19}$$

$$\phi_{\rm S}^{(1)} = \phi_{\rm I}^{(1)} + \phi_{\rm D}^{(1)} \sim H(z)\begin{cases} \mathrm{e}^{\mathrm{i}kx} + \sum_{n=0}^{M} R_n^{(1)} Y_n(y)\mathrm{e}^{-\mathrm{i}kxt_n} & \text{as } x \to -\infty \\ \sum_{n=0}^{M} T_n^{(1)} Y_n(y)\mathrm{e}^{\mathrm{i}kxt_n} & \text{as } x \to +\infty, \end{cases} \tag{A 20}$$

$$\phi_{\rm S}^{(2)} = \phi_{\rm I}^{(2)} + \phi_{\rm D}^{(2)} \sim H(z)\begin{cases} \sum_{n=0}^{M} T_n^{(2)} Y_n(y)\mathrm{e}^{-\mathrm{i}kxt_n} & \text{as } x \to -\infty \\ \mathrm{e}^{-\mathrm{i}kx} + \sum_{n=0}^{M} R_n^{(2)} Y_n(y)\mathrm{e}^{\mathrm{i}kxt_n} & \text{as } x \to +\infty, \end{cases} \tag{A 21}$$

where the coefficients $A_{in}^{\pm}$, $i = 1,\ldots,6$, $R_n^{(i)}$, $T_n^{(i)}$, $i = 1,2$, are to be determined.

The generalized hydrodynamic force on the body in the ith direction is given by $F_i(t) = \mathrm{Re}\{X_i \mathrm{e}^{-\mathrm{i}\omega t}\}$ where

$$X_i = -\mathrm{i}\rho\omega \int_{S_{\rm B}} \phi n_i \,\mathrm{d}S. \tag{A 22}$$

This can be written

$$X_i = -\sum_{j=1}^{6} U_j(-\mathrm{i}\omega M_{ij} + B_{ij}) \tag{A 23}$$

where $\boldsymbol{M}$ and $\boldsymbol{B}$ are real matrices ($\boldsymbol{M}$ is the added-mass matrix and $\boldsymbol{B}$ is the damping matrix). Thus

$$-\mathrm{i}\omega M_{ij} + B_{ij} = \mathrm{i}\rho\omega \int_{S_{\rm B}} \phi_j n_i \,\mathrm{d}S. \tag{A 24}$$

The exciting force in the ith direction on the body due to the scattering potential $\phi_{\rm S}^{(j)}$ is

$$X_{{\rm S}i}^{(j)} = -\mathrm{i}\rho\omega \int_{S_{\rm B}} \phi_{\rm S}^{(j)} n_i \,\mathrm{d}S, \qquad j = 1,2. \tag{A 25}$$

The starting point for the derivation of general identities involving the above quantities is Green's theorem which implies that if ψ_1 and ψ_2 are harmonic in a region surrounded by a closed surface S then

$$\int_S \left(\psi_1 \frac{\partial \psi_2}{\partial n} - \psi_2 \frac{\partial \psi_1}{\partial n} \right) \mathrm{d}S = 0. \tag{A 26}$$

In all the cases that follow ψ_1 and ψ_2 will have zero normal derivative on the walls and the bottom of the channel and will satisfy the free-surface boundary condition. Thus

$$\int_{S_B+S_\infty} \left(\psi_1 \frac{\partial \psi_2}{\partial n} - \psi_2 \frac{\partial \psi_1}{\partial n} \right) \mathrm{d}S = 0. \tag{A 27}$$

Here S_∞ is made up of the two regions $\{x = \pm X, -d < y < d, -h < z < 0\}$ and $|X|$ is large enough for the asymptotic forms for the potentials given in (A 19)–(A 21) to be valid.

We will begin by taking $\psi_1 = \phi_i$ and $\psi_2 = \phi_j$. It is straightforward to show that the contribution from S_∞ is zero. We make use of the fact that

$$\int_{-d}^{d} Y_n(y) Y_m(y)\, \mathrm{d}y = 2d\epsilon_n^{-1}\delta_{nm}, \tag{A 28}$$

where $\epsilon_0 = 1$ and $\epsilon_n = 2$ if $n \geqslant 1$. Using the body boundary condition then gives

$$\int_{S_B} \phi_i n_j\, \mathrm{d}S = \int_{S_B} \phi_j n_i\, \mathrm{d}S, \tag{A 29}$$

from which it follows that the matrices $\boldsymbol{M}$ and $\boldsymbol{B}$ are symmetric.

Next we take $\psi_1 = \phi_i$ and $\psi_2 = \overline{\phi_j}$. Here an overbar represents the complex conjugate and we have from (A19) that

$$\overline{\phi_j} \sim \overline{H(z)} \sum_{n=0}^{M} \overline{A_{in}^{\pm}} Y_n(y) \mathrm{e}^{\mp \mathrm{i}kxt_n} \qquad \text{as } x \to \pm\infty. \tag{A 30}$$

Thus

$$2\mathrm{i}\,\mathrm{Im} \int_{S_B} \phi_i n_j\, \mathrm{d}S = -\int_{S_\infty} \left(\phi_i \frac{\partial \overline{\phi_j}}{\partial n} - \phi_j \frac{\partial \overline{\phi_i}}{\partial n} \right) \mathrm{d}S$$

$$= -\int_{-h}^{0} |H(z)|^2\, \mathrm{d}z \left\{ \sum_{n=0}^{M} 4\mathrm{i}kd\epsilon^{-1} t_n (A_{in}^{+}\overline{A_{jn}^{+}} + A_{in}^{-}\overline{A_{jn}^{-}}) \right\}. \tag{A 31}$$

Now

$$\int_{-h}^{0} |H(z)|^2\, \mathrm{d}z = \frac{g^2}{\omega^2 \cosh^2 kh} \int_{-h}^{0} \cosh^2 k(z+h)\, \mathrm{d}z = \frac{gc_g}{\omega k}$$

where $c_g = \omega k^{-1}(1 + 2kh/\sinh 2kh)$ is the group velocity. It follows from (A 24) that

$$B_{ij} = 2\rho g d c_g \sum_{n=0}^{M} \epsilon_n^{-1} t_n (A_{in}^{+}\overline{A_{jn}^{+}} + A_{in}^{-}\overline{A_{jn}^{-}}). \tag{A 32}$$

In particular

$$B_{ii} = 2\rho g d c_g \sum_{n=0}^{M} \epsilon_n^{-1} t_n (|A_{in}^{+}|^2 + |A_{in}^{-}|^2). \tag{A 33}$$

If we use scattering potentials instead of radiation potentials in (A 27) further results can be obtained. Thus if we take $\psi_1 = \phi_S^{(1)}$ and $\psi_2 = \phi_S^{(2)}$ it is readily seen that the integral over the body surface gives no contribution and the integral over S_∞ leads to the simple result

$$T_0^{(1)} = T_0^{(2)}. \tag{A 34}$$

Taking $\psi_1 = \phi_S^{(1)}$ and $\psi_2 = \overline{\phi_S^{(1)}}$ leads to a result expressing the conservation of energy. First we note that

$$\overline{\phi_S^{(1)}} \sim \overline{H(z)} \begin{cases} e^{-ikx} + \sum_{n=0}^{M} \overline{R_n^{(1)}}\, Y_n(y) e^{ikxt_n} & \text{as } x \to -\infty \\ \sum_{n=0}^{M} \overline{T_n^{(1)}}\, Y_n(y) e^{-ikxt_n} & \text{as } x \to +\infty, \end{cases} \tag{A 35}$$

and then (A 27) implies that

$$\sum_{n=0}^{M} \epsilon_n^{-1} t_n (|R_n^{(1)}|^2 + |T_n^{(1)}|^2) = 1. \tag{A 36}$$

Similarly it can be shown that

$$\sum_{n=0}^{M} \epsilon_n^{-1} t_n (|R_n^{(2)}|^2 + |T_n^{(2)}|^2) = 1. \tag{A 37}$$

Note that if $M = 0$, since $T_0^{(1)} = T_0^{(2)}$, we have $|R_0^{(1)}| = |R_0^{(2)}|$.

Since the numbers n_i appearing in (A 10) are real it follows that the potential $\psi_2 = \phi_i - \overline{\phi_i}$ has zero normal derivative on the body boundary. Also

$$\psi_2 \sim H(z) \sum_{n=0}^{M} A_{in}^{\pm} Y_n(y) e^{\pm ikxt_n} - \overline{H(z)} \sum_{n=0}^{M} \overline{A_{in}^{\pm}}\, Y_n(y) e^{\mp ikxt_n} \qquad \text{as } x \to \pm\infty. \tag{A 38}$$

Using this function together with $\psi_1 = \phi_S^{(1)}$ gives

$$A_{i0}^{-} + \sum_{n=0}^{M} \epsilon_n^{-1} t_n (\overline{A_{in}^{-}} R_n^{(1)} + \overline{A_{in}^{+}} T_n^{(1)}) = 0. \tag{A 39}$$

With $\psi_1 = \phi_S^{(2)}$ we obtain

$$A_{i0}^{+} + \sum_{n=0}^{M} \epsilon_n^{-1} t_n (\overline{A_{in}^{+}} R_n^{(2)} + \overline{A_{in}^{-}} T_n^{(2)}) = 0. \tag{A 40}$$

These relations, which can be used both for the evaluation of coefficients and for numerical accuracy tests, are extensions of the Newman–Bessho relations (Newman 1976).

Finally, a relation can be obtained between the exciting force in the ith direction due to an incident wave and the amplitude of the radiated wave due to motion in the ith mode. Thus if we take $\psi_1 = \phi_i$ and $\psi_2 = \phi_D^{(1)}$ we obtain

$$\int_{S_B} \left(\phi_i \frac{\partial \phi_D^{(1)}}{\partial n} - \phi_D^{(1)} \frac{\partial \phi_i}{\partial n} \right) dS = 0. \tag{A 41}$$

Using (A 9) and (A 25) this can be rearranged to give

$$X_{Si}^{(1)} = i\rho\omega \int_{S_B} \left(\phi_i \frac{\partial \phi_I^{(1)}}{\partial n} - \phi_I^{(1)} \frac{\partial \phi_i}{\partial n} \right) dS. \tag{A 42}$$

Now if we use $\psi_1 = \phi_i$ and $\psi_2 = \phi_{\rm I}^{(1)}$ we obtain

$$\int_{S_{\rm B}} \left(\phi_i \frac{\partial \phi_{\rm I}^{(1)}}{\partial n} - \phi_{\rm I}^{(1)} \frac{\partial \phi_i}{\partial n} \right) {\rm d}S = -4{\rm i}kdAA_{i0}^{+} \int_{-h}^{0} [H(z)]^2 \, {\rm d}z, \qquad (A\,43)$$

which when combined with (A 42) gives

$$X_{Si}^{(1)} = -4d\rho g c_{\rm g} AA_{i0}^{+}. \qquad (A\,44)$$

Similarly

$$X_{Si}^{(2)} = -4d\rho g c_{\rm g} AA_{i0}^{-}. \qquad (A\,45)$$

These relations are extensions of the Haskind relations, Haskind (1959).

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